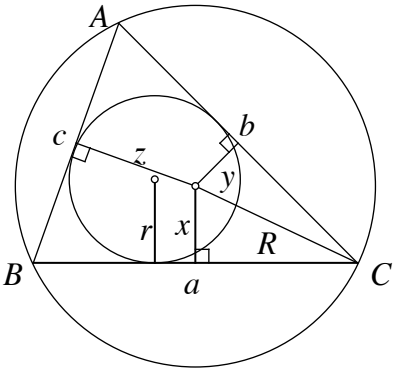
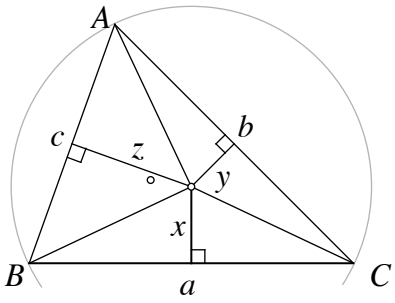
Proof Without Words: Carnot's Theorem for Acute Triangles

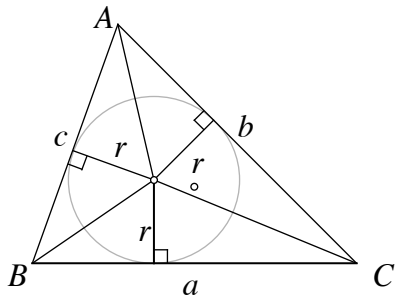
In an acute triangle ABC, the sum of the distances x, y, z from the circumcenter to the sides is equal to the sum of the inradius r and circumradius R, that is, $x + y + z = R + r$.



Lemma 1. $ax + by + cz = r(a + b + c)$.

Proof.

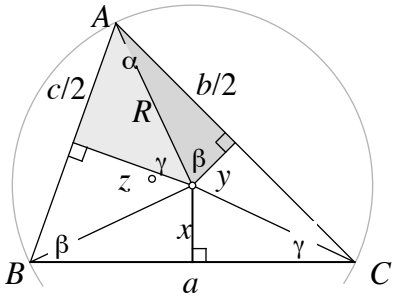


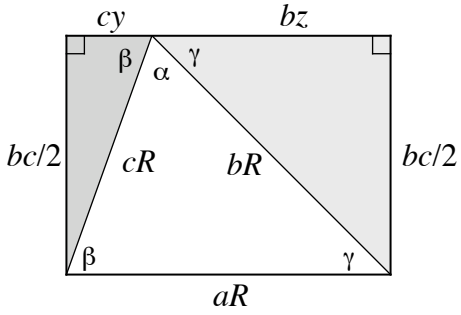


$$\frac{1}{2}ax + \frac{1}{2}by + \frac{1}{2}cz = \frac{1}{2}r(a + b + c), \quad \therefore ax + by + cz = r(a + b + c).$$

Lemma 2. $cy + bz = aR$; $az + cx = bR$; $bx + ay = cR$.

Proof (that $cy + bz = aR$; the other two relationships are proved analogously).





Proof of Carnot's Theorem.

$$\begin{aligned}(a + b + c)(x + y + z) &= (ax + by + cz) + (cy + bz) + (az + cx) + (bx + ay)\\ &= r(a + b + c) + (a + b + c)R\\ &= (a + b + c)(R + r)\\ \therefore x + y + z &= R + r\end{aligned}$$

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