The Maclaurin Inequality and Rectangular Boxes

CLAUDI ALSINA
Universitat Politècnica de Catalunya Barcelona, Spain
claudio.alsina@upc.edu

ROGER B. NELSEN
Lewis & Clark College
Portland, OR
nelsen@lclark.edu

In a recent article [1], Ben-Ari and Conrad introduced (or reintroduced) readers of this MAGAZINE to Maclaurin's inequality for n nonnegative numbers. A special case is $n = 3$, when the Maclaurin inequality for nonnegative numbers x, y, and z is

$$\frac{x+y+z}{3} \geqslant \sqrt{\frac{xy+xz+yz}{3}} \geqslant \sqrt[3]{xyz} \tag{1}$$

with equality throughout if and only if $x = y = z$. As noted in [1], the arithmetic mean–geometric mean inequality for three nonnegative numbers is (1) without the middle term.

Let x, y, and z be the dimensions of a rectangular box (parallelepiped or cuboid). Three quantities of interest associated with the box are its volume $V = xyz$, the total area $F = 2(xy + xz + yz)$ of its six faces, and the total length $E = 4(x + y + z)$ of its 12 edges. Thus, (1) relates these three numbers as

$$\frac{E}{12} \geqslant \sqrt{\frac{F}{6}} \geqslant \sqrt[3]{V} \tag{2}$$

with equality throughout if and only if the box is a cube.

In this note, we present a visual proof of (1) using areas of squares and rectangles and volumes of cubes and boxes after rewriting (1) as

$$(x+y+z)^2 \geqslant 3(xy+xz+yz) \quad \text{and} \quad xy+xz+yz \geqslant 3(xyz)^{2/3}. \tag{3}$$

We assume without loss of generality that $x \geqslant y \geqslant z$ and begin with a simple lemma.

Lemma 1. *If x, y, and z are nonnegative, then $x^2 + y^2 + z^2 \geqslant xy + xz + yz$.*

Proof. See Figure 1, where we compare the areas of three squares to the areas of three rectangles. ∎

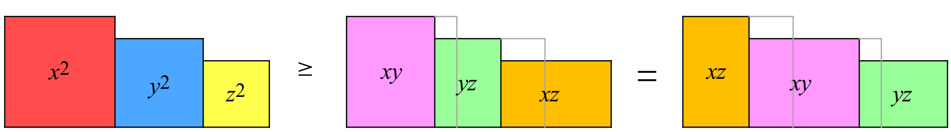

Figure 1

Math. Mag. **90** (2017) 228–230. doi:10.4169/math.mag.90.3.228.
MSC: Primary 26D07

Theorem 1. *If x, y, and z are nonnegative, then* $(x + y + z)^2 \geqslant 3(xy + xz + yz)$.

Proof. See Figure 2, where we use Lemma 1 to establish the inequality. ■

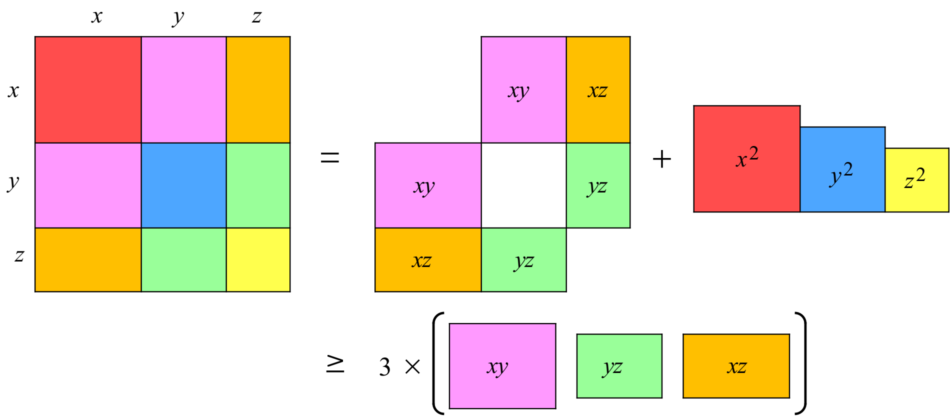

Figure 2

If we let $a^3 = xy$, $b^3 = xz$, and $c^3 = yz$, then $abc = (xyz)^{2/3}$ so that the second inequality in (3) is equivalent to $a^3 + b^3 + c^3 \geqslant 3abc$, which we now prove.

Theorem 2. *If a, b, and c are nonnegative, then* $a^3 + b^3 + c^3 \geqslant 3abc$.

Proof. Let $a \geqslant b \geqslant c$. Comparing volumes of cubes and boxes in Figure 3 (a three-dimensional version of Figure 1), part (a) shows that $a^3 + b^3 + c^3 \geqslant a^2b + b^2c + c^2a$ and part (b) shows that $a^3 + b^3 + c^3 \geqslant a^2c + b^2a + c^2b$. The same two inequalities hold for other orders of a, b, and c.

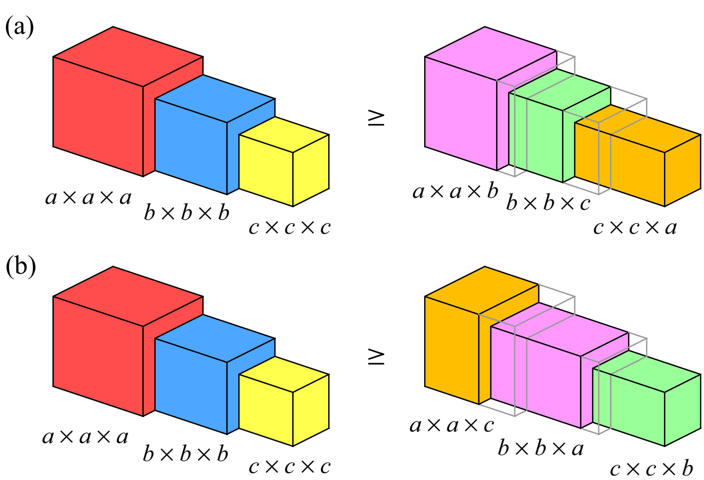

Figure 3

Adding the two inequalities yields

$$2(a^3 + b^3 + c^3) \geqslant a^2b + b^2c + c^2a + a^2c + b^2a + c^2b.$$

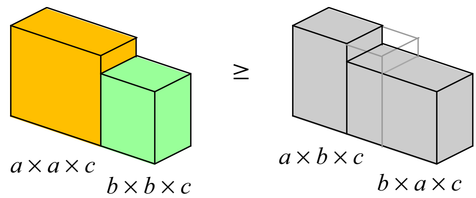

Figure 4

In Figure 4, again comparing volumes, we see that $a^2c + b^2c \geqslant 2abc$, and similar figures show that $a^2b + bc^2 \geqslant 2abc$ and $ab^2 + ac^2 \geqslant 2abc$.

Hence, we have $2(a^3 + b^3 + c^3) \geqslant 6abc$, proving Theorem 2. ■

Lemma 1 and a figure similar to Figure 2 can be used to establish the inequality $3(x^2 + y^2 + z^2) \geqslant (x + y + z)^2$, from which it follows that, if $d = \sqrt{x^2 + y^2 + z^2}$ is the length of the space diagonal of the rectangular box, then $d/\sqrt{3} \geq E/12$.

There are interpretations similar to (2) of the terms of Maclaurin's inequality for $n \geqslant 4$ in terms of n-dimensional volumes of *hyperrectangles* (or *n-orthotopes*) and their lower-dimensional facets. We leave the details to the interested reader.

REFERENCE

1. I. Ben-Ari, K. Conrad, Maclaurin's inequality and a generalized Bernoulli inequality, *Math. Mag.*, **87** (2014), 14–24.

Summary. We interpret the terms of Maclaurin's inequality for three nonnegative numbers as the volume, total face area, and total side length of a rectangular box and provide a visual proof of the inequality.

CLAUDI ALSINA (MR Author ID 25110) is professor of mathematics at Universitat Politècnica de Catalunya.

ROGER B. NELSEN (MR Author ID 237909) is professor emeritus at Lewis & Clark College, where he taught mathematics and statistics for 40 years.

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